

Conjugate variables in finite phase plane

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2005 J. Phys. A: Math. Gen. 38 L389

(<http://iopscience.iop.org/0305-4470/38/21/L03>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.66

The article was downloaded on 02/06/2010 at 20:14

Please note that [terms and conditions apply](#).

LETTER TO THE EDITOR

Conjugate variables in finite phase plane**A Mann, M Revzen and J Zak**

Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel

E-mail: ady@physics.technion.ac.il, revzen@physics.technion.ac.il and zak@physics.technion.ac.il

Received 30 March 2005

Published 10 May 2005

Online at stacks.iop.org/JPhysA/38/L389**Abstract**

We construct two pairs of quasicordinates and quasimomenta in a finite phase plane, which form sets of conjugate variables. In such a plane the coordinate x is quantized with a step c , and the momentum p with a step $\frac{2\pi}{Mc}$, where Mc is the size of the phase plane in the x -direction. The construction depends crucially on the possibility of writing $M = M_1 M_2$ with M_1 and M_2 relatively prime. The conjugate variables are applied to Harper-like Hamiltonians. It is shown how to design physical systems with energy spectra containing any desired number of discrete energy levels, say M_1 , each of them having a prescribed degeneracy M_2 .

PACS numbers: 73.20.Dx, 02.20.Df, 03.65.-w

A general theory of quantum mechanics in a finite phase plane was developed by Schwinger [1]. It is achieved by applying boundary conditions on the coordinate x for a wavefunction $\psi(x)$ and on its Fourier transform $F(p)$

$$\psi(x + Mc) = \psi(x); \quad F\left(p + \hbar \frac{2\pi}{c}\right) = F(p), \quad (1)$$

where M is an integer and c is a constant. As a consequence of these boundary conditions, the coordinate x and the momentum p are quantized and they assume the following discrete values:

$$x = sc, \quad s = 1, \dots, M; \quad p = \hbar \frac{2\pi}{Mc} t, \quad t = 1, \dots, M. \quad (2)$$

In this framework the operators x and p are replaced by the exponential operators $\tau\left(\frac{2\pi}{Mc}\right)$ and $T(c)$

$$\tau\left(\frac{2\pi}{Mc}\right) = \exp\left(ix \frac{2\pi}{Mc}\right), \quad T(c) = \exp\left(\frac{i}{\hbar} pc\right). \quad (3)$$

From equations (2) and (3), it follows that

$$\left[\tau \left(\frac{2\pi}{Mc} \right) \right]^M = [T(c)]^M = 1. \quad (4)$$

Quantum mechanics in a finite phase plane finds applications in a great variety of areas such as the quantum Hall effect [2], quantum maps [3], Landau levels in a magnetic field, von Neumann lattices [4] and quantum computing [5]. For a review article see [6].

We now assume that

$$M = M_1 M_2 \quad (5)$$

with M_1 and M_2 relatively prime numbers. This assumption is crucial for what follows. Following equation (5), we introduce two constants

$$a = M_1 c \quad \text{and} \quad b = M_2 c, \quad (6)$$

and define two kq -representations based on the two complete sets of commuting operators [7]

$$\tau \left(\frac{2\pi}{a} \right) = e^{ix \frac{2\pi}{a}}, \quad T(a) = e^{\frac{i}{\hbar} p a}; \quad \tau \left(\frac{2\pi}{b} \right) = e^{ix \frac{2\pi}{b}}, \quad T(b) = e^{\frac{i}{\hbar} p b}. \quad (7)$$

We have

$$\left[\tau \left(\frac{2\pi}{a} \right), T(a) \right] = \left[\tau \left(\frac{2\pi}{b} \right), T(b) \right] = 0 \quad (7a)$$

but

$$\left. \begin{aligned} T(a) \tau \left(\frac{2\pi}{b} \right) &= \tau \left(\frac{2\pi}{b} \right) T(a) \exp \left(2\pi i \frac{M_1}{M_2} \right) \\ \text{and} \\ T(b) \tau \left(\frac{2\pi}{a} \right) &= \tau \left(\frac{2\pi}{a} \right) T(b) \exp \left(2\pi i \frac{M_2}{M_1} \right) \end{aligned} \right\} \quad (7b)$$

It therefore follows that the operators $T(a)$ and $\tau \left(\frac{2\pi}{a} \right)$ and their powers form a set of M commuting operators. The same can be said about the operators $T(b)$ and $\tau \left(\frac{2\pi}{b} \right)$. This means that the operators in equations (7) together with all their distinct products lead to M^2 distinct operators which replace the M^2 operators in equation (3).

Let us now define the eigenvectors and eigenvalues of each of the two commuting sets of operators in equation (7):

$$\tau \left(\frac{2\pi}{a} \right) |k, q\rangle = e^{iq \frac{2\pi}{a}} |k, q\rangle; \quad T(a) |k, q\rangle = e^{ika} |k, q\rangle \quad (8)$$

$$\tau \left(\frac{2\pi}{b} \right) |K, Q\rangle = e^{iQ \frac{2\pi}{b}} |K, Q\rangle; \quad T(b) |K, Q\rangle = e^{iKb} |K, Q\rangle \quad (9)$$

where $|k, q\rangle$ and $|K, Q\rangle$ are, respectively, the eigenvectors of the pairs of commuting operators $\tau \left(\frac{2\pi}{a} \right), T(a)$ and $\tau \left(\frac{2\pi}{b} \right), T(b)$ in equation (7). In the x -representation the eigenvectors in equations (8) and (9) are [4, 7]

$$\langle x | k, q \rangle = \frac{1}{\sqrt{M_2}} \sum_{s=1}^{M_2} \exp(iks a) \Delta(x - q - sa), \quad (10)$$

$$\langle x | K, Q \rangle = \frac{1}{\sqrt{M_1}} \sum_{t=1}^{M_1} \exp(iKtb) \Delta(x - Q - tb), \quad (11)$$

Table 1. Solutions of equation (16) for $M = 10, M_1 = 2, M_2 = 5$ (see text).

s	1	1	2	2	3	3	4	4	5	5
t	1	2	1	2	1	2	1	2	1	2
r	3	8	1	6	9	4	7	2	5	10

where $\Delta(x)$ is 1 when x is a multiple of Mc , and is zero otherwise. In equations (10) and (11) the variables k, q, K and Q assume the following values:

$$k = \frac{2\pi}{Mc} f, \quad f = 1, \dots, M_2, \quad q = gc, \quad g = 1, \dots, M_1 \tag{12}$$

$$K = \frac{2\pi}{Mc} f', \quad f' = 1, \dots, M_1, \quad Q = g'c, \quad g' = 1, \dots, M_2. \tag{13}$$

With this information at hand we can find the probability of measuring the variables k and q , when the particle is in the state $|K, Q\rangle$. For this we first find $\langle k, q|K, Q\rangle$ (we use equations (10) and (11))

$$\begin{aligned} \langle k, q|K, Q\rangle &= \sum_{x=0}^{Mc} \langle k, q|x\rangle \langle x|K, Q\rangle \\ &= \frac{1}{\sqrt{M_1 M_2}} \sum_{s=1}^{M_2} \sum_{t=1}^{M_1} \exp(-iks a + iKtb) \Delta(Q + tb - q - sa). \end{aligned} \tag{14}$$

Because of the Δ -function, $\langle k, q|K, Q\rangle$ in equation (14) does not vanish only when $Q - q + tb - sa = 0$, modulo Mc . Let us assume that $Q = mc, q = nc$. Having in mind that $a = M_1c$ and $b = M_2c$ (equation (6)), the equation

$$Q - q + tb - sa = 0, \quad \text{modulo } Mc \tag{15}$$

becomes

$$tM_2 - sM_1 = r, \quad \text{modulo } M, \tag{16}$$

where $r = n - m$, and M_1 and M_2 are relatively prime as was pointed out before. It can be shown that for each value of r between 1 and M there is a single pair (s, t) that solves equation (16). This means that the Diophantine equation (16) has exactly M different triples (r, s, t) that solve it with r running from 1 to M, s from 1 to M_2 and t from 1 to M_1 . An example of such triples is given in table 1 for $M = 10, M_1 = 2$ and $M_2 = 5$.

Let us now come back to equation (14) for $\langle k, q|K, Q\rangle$. For any two coordinates q and Q the Δ -function in equation (14) will be 1, when equation (15) (or equation (16)) is satisfied, and zero otherwise. From here it follows that the double sum in equation (14) will reduce to one term, and we find for $\langle k, q|K, Q\rangle$

$$\langle k, q|K, Q\rangle = \frac{1}{\sqrt{M_1 M_2}} \exp(-iks a + iKtb), \tag{17}$$

where $q - Q = rc$, and r, s and t satisfy equation (16). This actually means that $\langle k, q|K, Q\rangle$ does not depend on q and Q explicitly, and the dependence on k and K is given by just the phase in equation (17). One has to keep in mind, however, that s and t in the phase in equation (17) are determined by r in equation (16). From equation (17) it follows that

$$|\langle k, q|K, Q\rangle|^2 = \frac{1}{M_1 M_2} = \frac{1}{M}. \tag{18}$$

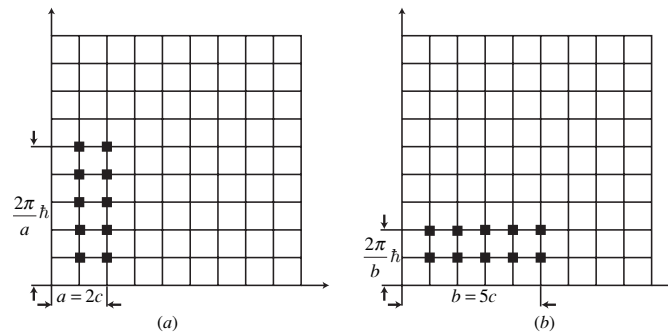


Figure 1. (a) Eigenvalues of the a -set operators. (b) Eigenvalues of the b -set operators.

This result shows that when the particle is in the eigenstate $|K, Q\rangle$ of the commuting operators $T(b)$ and $\tau\left(\frac{2\pi}{b}\right)$ (see equation (7)), the probability of measuring k and q does not depend on k and q . The same can be said about measuring K and Q in the eigenstate $|k, q\rangle$. We can therefore claim that the two sets of commuting operators in equation (7) are conjugate (like the operators x and p).

A very interesting property of conjugate operators is as follows. We call the commuting sets of pairs of operators in equation (7) the a and the b -sets, respectively. We can show that when the a -set operators in equation (7) operate on the eigenvectors of the b -set, the eigenvalues of these eigenvectors are shifted. Let us first find the eigenvalues of the vectors $T(a)|K, Q\rangle$. We have, using the first equation in equations (7b), and (9)

$$\tau\left(\frac{2\pi}{b}\right) T(a)|K, Q\rangle = e^{i(Q-a)\frac{2\pi}{b}} T(a)|K, Q\rangle, \quad (19)$$

where $a = M_1c$ (see equation (6)). Equation (19) shows that applying $T(a)$ to the eigenvector $|K, Q\rangle$ results in an eigenvector corresponding to the eigenvalue $Q - a$. Similarly, if we apply $T^2(a)$ to $|K, Q\rangle$ we will get an eigenvector corresponding to $Q - 2a$, and so on. Applying all the operators $T^\ell(a)$ with $\ell = 1, \dots, M_2$ to $|K, Q\rangle$, we will obtain M_2 eigenvectors $T^\ell(a)|K, Q\rangle$ corresponding to the eigenvalues $Q - \ell a$. This means that starting with a fixed Q , and ℓ assuming the values $\ell = 1, \dots, M_2$, $Q - \ell a$ will assume all possible eigenvalues of $\tau\left(\frac{2\pi}{b}\right)$ in equation (13) (or equation (19)). This is easy to see because assuming $Q = g'c$ (see equation (13)), $Q - \ell a = g'c - \ell M_1c$. Bearing in mind that M_1 and M_2 are relatively prime, the latter equation modulo M will lead to all possible values of Q in equation (13). In a similar way one can show that applying $\tau^m\left(\frac{2\pi}{a}\right)$ with $m = 1, \dots, M_1$, to the eigenvector $|K, Q\rangle$, one obtains all M_1 eigenvectors with eigenvalues $K = \frac{2\pi}{Mc} f'$, as given in equation (13). Therefore we reach the following expected result for conjugate operators: when applying the M operators of the a -set $[T(a)$ and $\tau\left(\frac{2\pi}{a}\right)]$ to a fixed eigenvector $|K, Q\rangle$ of the b -set $[T(b)$ and $\tau\left(\frac{2\pi}{b}\right)]$, we obtain all the eigenvectors of the b -set. And vice versa, when applying the M operators of the b -set to a fixed eigenvector $|k, q\rangle$ of the a -set, we restore all the eigenvectors of the a -set. In figures 1(a) and (b) we plot the eigenvalues of the a -set and the b -set, respectively, for the example considered above, $M = 10$, $M_1 = 2$, $M_2 = 5$ (see table 1).

Alternatively, one may say that when all the $M = 10$ a -set operators are applied to a single eigenvector of the b -set this will lead to ten different eigenvectors of the b -set, corresponding to the ten eigenvalues in figure 1(b). And, vice versa, when the $M = 10$ b -set operators are applied to a single eigenvector of the a -set, we will obtain the ten different eigenvectors of the a -set corresponding to the ten eigenvalues in figure 1(a). What is important to stress here is that all the eigenvalues are included in this process, which is a consequence of the a -set and

b -set being conjugate. For either of these sets, their eigenvalues lie in a unit cell of area h , the Planck constant. The covering of all the eigenvalues in the unit cells of the phase plane is reminiscent of ergodicity in statistical physics.

Having established the conjugacy of the two sets of operators in equation (7), one can now discuss the wavefunctions $C^{(a)}(k, q)$ and $C^{(b)}(K, Q)$ in the kq and KQ -representations. We shall use the following notation for the different representations of the state $|\psi\rangle$:

$$\begin{aligned} \psi(x) &= \langle x|\psi\rangle, & F(p) &= \langle p|\psi\rangle, \\ C^{(a)}(k, q) &= \langle k, q|\psi\rangle, & C^{(b)}(K, Q) &= \langle K, Q|\psi\rangle. \end{aligned} \tag{20}$$

For finding relations between the functions in equation (20), it is convenient to employ the various representations of the unit operator I

$$\sum_x |x\rangle\langle x| = \sum_p |p\rangle\langle p| = \sum_{k,q} |k, q\rangle\langle k, q| = \sum_{K,Q} |K, Q\rangle\langle K, Q| = I. \tag{21}$$

In the finite phase plane (see equation (1)) the Fourier transform $F(p)$ of $\psi(x)$ is

$$F(p) = \langle p|\psi\rangle = \sum_x \langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{M}} \sum_x \exp\left(-\frac{i}{h}xp\right) \psi(x), \tag{22}$$

where x and p are given in equation (2), and the result was used that the eigenstate $|p\rangle$ of $T(c)$ (equation (3)) in the x -representation is [1]

$$\langle x|p\rangle = \frac{1}{\sqrt{M}} \exp\left(\frac{i}{h}px\right). \tag{23}$$

The absolute value of the square of $\langle x|p\rangle$ is a constant, $|\langle x|p\rangle|^2 = \frac{1}{M}$. This is a consequence of the fact that the operators in equation (3), $\tau\left(\frac{2\pi}{Mc}\right)$ and $T(c)$ are conjugate. The relation between $C^{(a)}(k, q)$ and $\psi(x)$ (see equation (20)) is well known [4]

$$C^{(a)}(k, q) = \langle k, q|\psi\rangle = \sum_x \langle k, q|x\rangle\psi(x) = \frac{1}{\sqrt{M_2}} \sum_{s=1}^{M_2} \exp(ikas)\psi(q - sa). \tag{24}$$

It is of interest to find also an equation, similar to equation (22), connecting the functions $C^{(b)}(K, Q)$ and $C^{(a)}(k, q)$.

We have, by using equation (14)

$$C^{(b)}(K, Q) = \sum_{k,q} \langle K, Q|k, q\rangle\langle k, q|\psi\rangle = \frac{1}{\sqrt{M}} \sum_k \sum_{t=1}^{M_1} \exp(-iKtb)C^{(a)}(k, Q + tb). \tag{25}$$

In deriving equation (25), we used the fact that for each fixed difference $q - Q = rc$, the Diophantine equation (16) determines uniquely s and t . Attention should be drawn to the summation over q in equation (25). As a consequence of this summation, the function $C^{(a)}(k, Q + tb - sa)$ appears. From the periodicity conditions, it follows that $C^{(a)}(k, Q + tb - sa) = \exp(-iksa)C^{(a)}(k, Q + tb)$. This explains the disappearance of the factor $\exp(iksa)$ from the expression $\langle K, Q|k, q\rangle$ of equation (14). Similarly, one finds

$$C^{(a)}(k, q) = \frac{1}{\sqrt{M}} \sum_K \sum_{s=1}^{M_2} \exp(-iksa)C^{(b)}(K, q + sa). \tag{26}$$

Equations (25) and (26) can be looked upon as analogues of the Fourier transform in equation (22). As an example of using equation (25), let us consider the case $M = 10$, $M_1 = 2$, $M_2 = 5$, and

$$C^{(a)}(k, q) = \frac{1}{\sqrt{M}}, \quad \text{for any } k, \quad \text{and } 0 < q \leq M_1c. \tag{27}$$

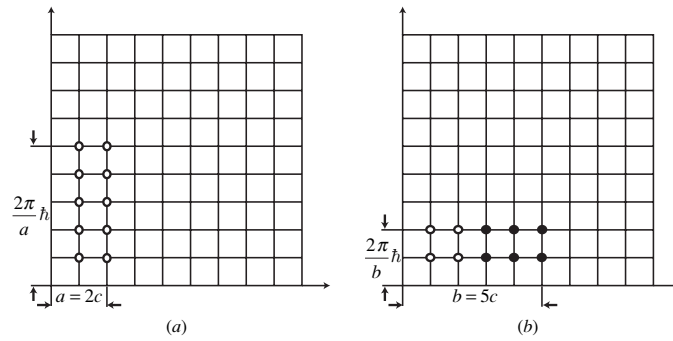


Figure 2. (a) Plot of $|C^{(a)}(k, q)|^2$ in equation (27). All the ten points in the unit cell have intensity of $\frac{1}{10}$. (b) Plot of $|C^{(b)}(K, Q)|^2$ in equations (28), (29). The four open circles in the unit cell have intensity of $\frac{1}{4}$. The six black circles have intensity zero.

For $K = \frac{2\pi}{10c}$, $Q = c$, we have for $C^{(b)}\left(\frac{2\pi}{10c}, c\right)$ in equation (25)

$$C^{(b)}\left(\frac{2\pi}{10c}, c\right) = \frac{1}{10} \sum_{f=1}^5 \left[\exp\left(-i\frac{2\pi}{10c}5c + i\frac{2\pi}{10c}f4c\right) + \exp\left(-i\frac{2\pi}{10c}10c + i\frac{2\pi}{10c}f10c\right) \right] = \frac{1}{2}. \quad (28)$$

In obtaining the result in equation (28), we used the fact that $C^{(a)}(k, 6c) = \exp(ik4c)C^{(a)}(k, 2c)$ and $C^{(a)}(k, 11c) = \exp(ik10c)C^{(a)}(k, c)$, which follows from using the periodic boundary conditions on $C^{(a)}(k, q)$. Similarly, we find

$$C^{(b)}\left(\frac{2\pi}{10c}, 2c\right) = C^{(b)}\left(\frac{2\pi}{5c}, c\right) = C^{(b)}\left(\frac{2\pi}{5c}, 2c\right) = \frac{1}{2}, \quad (29)$$

and zero for all other components of $C^{(b)}(K, Q)$. Since the norm of $C^{(a)}(k, q)$ in equation (27) is 1, one has to expect that also the norm of $C^{(b)}(K, Q)$ will be 1, which is confirmed by the results of equations (28) and (29). A graphical description of the results in equations (27)–(29) is given in figures 2(a) and (b). These figures show that the function $C^{(a)}(k, q)$, which is delocalized in the kq -representation (equal intensity at all points in the unit cell), becomes localized in the KQ -representation. This feature is not surprising in view of the fact that the kq - and KQ -representations are conjugate. One can actually show that for the $C^{(a)}(k, q)$ function in equation (27), for $M_1 = 2$ and for any large odd M_2 , the resulting $C^{(b)}(K, Q)$ is like in figure (2b) with $|C^{(b)}(K, Q)|^2 = \frac{1}{4}$ at the four open circles, and zero everywhere else. As was pointed out above, $C^{(a)}(k, q)$ and $C^{(b)}(K, Q)$ can be considered as Fourier transforms of each other.

An immediate natural application of the conjugate operators developed above is the active area of Harper-like Hamiltonians [8]. These are Hamiltonians which are periodic both in the momentum p and coordinate x . As an elementary example we consider a Hamiltonian which is a function of the operators $T(a)$ and $\tau\left(\frac{2\pi}{b}\right)$ (see equations (7))

$$H = H \left[T(a), \tau\left(\frac{2\pi}{b}\right) \right]. \quad (30)$$

This Hamiltonian is periodic in momentum p with period $\hbar\frac{2\pi}{a}$, and in coordinate x with period b , which means that the operators $T(b)$ and $\tau\left(\frac{2\pi}{a}\right)$ commute with H in equation (30). Bearing in mind that $T(b)$ and $\tau\left(\frac{2\pi}{a}\right)$ do not commute, we can label the eigenstates of H by either

$|\epsilon, K\rangle$ or $|\epsilon, q\rangle$ where ϵ is the eigenvalue of H , K the eigenvalue of $T(b)$ and q of $\tau(\frac{2\pi}{a})$. Let us choose the labelling $|\epsilon, K\rangle$ and write (See equation (9)):

$$H|\epsilon, K\rangle = \epsilon|\epsilon, K\rangle, \quad T(b)|\epsilon, K\rangle = \exp(iKb)|\epsilon, K\rangle. \quad (31)$$

Since $\tau(\frac{2\pi}{a})$ commutes with H , this means that also $\tau(\frac{2\pi}{a})|\epsilon, K\rangle$ is an eigenvector of H with the same eigenvalue ϵ . However, as was shown above (see text following equation (19)), when applying $\tau^m(\frac{2\pi}{a})$ with $m = 1, \dots, M_1$ to the eigenvector of $T(b)$, one obtains M_1 eigenvectors of $T(b)$ with eigenvalues $K = \frac{2\pi}{Mc}f'$, as given in equation (13). This proves that the eigenvalue ϵ of the Hamiltonian H in equation (31) is M_1 -fold degenerate.

As a simple example let us consider the Harper-like Hamiltonian

$$H = V_1 \cos\left(\frac{p}{\hbar}b\right) + V_2 \cos\left(x\frac{2\pi}{a}\right), \quad (32)$$

where V_1 and V_2 are constants. In the kq -representation (the a -set) the eigenvalue equation for the Hamiltonian in equation (32) is

$$\frac{V_1}{2} [C(k, q + b) + C(k, q - b)] + V_2 \cos\left(\frac{2\pi}{a}q\right) C(k, q) = \epsilon C(k, q). \quad (33)$$

From what was said above, this equation has M_1 eigenvalues ϵ , each of them M_2 -fold degenerate. It is easy to solve it for our example of $M = 10, M_1 = 2$ and $M_2 = 5$. In this case we have ten points in the finite phase plane (see figure 1(b)), and respectively we have to solve ten linear homogeneous equations for the ten unknowns $C(k, q)$. However, because of the 5-fold degeneracy of each energy level, it is sufficient to solve two equations with two unknowns, which we choose as follows (See equation (33))

$$\begin{aligned} \frac{V_1}{2} [e^{ik_1 2a} C(k_1, q_2) + e^{ik_1 2a} C(k_1, q_2)] + V_2 \cos\left(\frac{2\pi}{a}q_1\right) C(k_1, q_1) &= \epsilon C(k_1, q_1) \\ \frac{V_1}{2} [e^{ik_1 3a} C(k_1, q_1) + e^{ik_1 3a} C(k_1, q_1)] + V_2 \cos\left(\frac{2\pi}{a}q_2\right) C(k_1, q_2) &= \epsilon C(k_1, q_2) \end{aligned} \quad (34)$$

where we used the periodicity conditions of the kq -function $C(k, q)$ [7]

$$C(k, q) = C\left(k + \frac{2\pi}{a}, q\right) = \exp(-ika)C(k, q + a). \quad (35)$$

The notation in equation (34) is (see equation (12)): $k_1 = \frac{2\pi}{c}, q_1 = c, q_2 = 2c; a = 2c$, and we have used the fact that in our example $b = 5c$. By using these values, equation (34) becomes

$$V_1 e^{i\frac{4\pi}{5}} C_2 - V_2 C_1 = \epsilon C_1 \quad V_1 e^{-i\frac{4\pi}{5}} C_1 + V_2 C_2 = \epsilon C_2, \quad (36)$$

where $C_1 \equiv C(k_1, q_1)$ and $C_2 \equiv C(k_1, q_2)$ (see equation (12)). Solving equation (36) we find the eigenvalues ϵ and the ratio of C_1 and C_2

$$\epsilon_{1,2} = \pm\sqrt{V_1^2 + V_2^2}, \quad C_2^{(1,2)} = \frac{\epsilon_{1,2} + V_2}{V_1 \exp\left(i\frac{4\pi}{5}\right)} C_1^{(1,2)}, \quad (37)$$

where the superscripts in the wavefunction denote the two solutions corresponding to ϵ_1 and ϵ_2 . The other eight solutions of equations (34) will have the same energies $\epsilon_{1,2}$, while the wavefunctions will have the same expression as in equation (37), but with the exponents in the denominator $\exp\left(i\frac{4\pi}{5}m\right)$ with $m = 2, 3, 4, 5$. This corresponds to our general result that the solutions for each ϵ are 5-fold degenerate. It is interesting to point out that the solution in equation (37) also solves equation (33) for $M_1 = 2$ and any large odd M_2 . The energies are identical with these in equation (37) while for the wavefunctions we will have

in the denominator $\exp\left[i\frac{\pi}{M_2}(M_2 - 1)m\right]$ with $m = 1, 2, \dots, M_2$. This corresponds to a M_2 -degeneracy of each energy level as predicted by symmetry. Similarly, one can show that for $M = 2M_2$, where M_2 is odd ($a = 2c, b = M_2c$), the spectrum of the Harper-like Hamiltonian

$$H = V_1 \cos\left(\frac{p}{\hbar}a\right) + V_2 \cos\left(x\frac{2\pi}{b}\right) \quad (38)$$

will consist of M_2 discrete energy levels, each of them being doubly degenerate.

From the symmetry analysis and the explicit examples above, it follows that the Hamiltonians in equations (32) and (38) can be used in the design of physical systems with desired energy spectra, both from the point of view of the number of energy levels, and of their degeneracy. This can be achieved by the appropriate choice of M_1 and M_2 in equation (5). For example, when $M = 2 \times 3 \times 5 = 30$, there are three different choices for M_1 and M_2 , $(M_1, M_2) = (2, 15), (3, 10)$ and $(5, 6)$. This design of energy spectra might turn out to be of use in today's nanotechnology.

Acknowledgments

MR acknowledges with thanks informative discussions with Professors B G Englert and Y Aharonov.

References

- [1] Schwinger J 1960 *Proc. Natl Acad. Sci.* **46** 570
- [2] Wen X G and Niu Q 1990 *Phys. Rev. B* **41** 9377
- [3] Leboeuf P, Kurchau J, Feingold M and Arovass D P 1992 *Chaos* **2** 125
- [4] Zak J 1989 *J. Math. Phys.* **30** 1591
- [5] Bartlett S D, de Guise H and Sanders B C 2002 *Phys. Rev. A* **65** 052316
- [6] Vourdas A 2004 *Rep. Prog. Phys.* **67** 267
- [7] Zak J 1967 *Phys. Rev. Lett.* **19** 1385
Zak J 1970 *Phys. Today* **23** 51
- [8] Harper P G 1955 *Proc. Phys. Soc. A* **68** 874
Pankrashkin K 2004 *J. Phys. A: Math. Gen.* **37** 11681 and references therein